ARIEL UNIVERSITY

אוניברסיטת אריאל בשומרון

Foundations of Modern Cryptography

Edward A. Hirsch

https://edwardahirsch.github.io/edwardahirsch

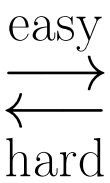
Neapolis University Pafos

Lecture 2: October 9, 2024

Lecture Plan

- One-Way Functions
  - Warm-up: toy example, worse-case one-way functions
  - One-way functions and families
  - A universal one-way function
- Trapdoor functions (families)

Worst-case "one-way functions"



«Naïve approach»

no randomness

- «easy»: $\in$ **fP** (computable in deterministic polynomial time),
- «hard»: $\notin$ **fP** (or **FNP**) (not computable in ...).

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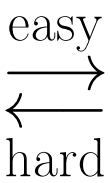

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Proof.

Let Φ be a Boolean formula,
let $A \in \{0,1\}^n$ be a Boolean assignment to its n variables.
Define

$$f(\Phi, A) = \begin{cases} (\Phi, 1^n), & \text{if } \Phi[A] = \texttt{True}, \\ (\Phi, 0^n), & \text{otherwise.} \end{cases}$$

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Exercise

Prove the converse to our "worst-case theorem": If $\exists f\ (f \in \boldsymbol{f}\mathbf{P} \wedge f^{-1} \notin \mathbf{FNP})$, then $\mathbf{P} \neq \mathbf{NP}$.

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  Yes, a lot of them: there are 2^{2^n} functions $\{0,1\}^n \to \{0,1\}$ and $2^{O\left(n \cdot 2^{\sqrt{n}}\right)}$ circuits of size $\leqslant 2^{\sqrt{n}}$.

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- Are there one-way functions?

  Hmm... at least, not many: a randomly generated function has a great chance to be hard in both directions, so not one-way.

Exercise

Almost all length-preserving functions have "similar" circuit complexity in both directions.

Families of one-way functions

A single one-way function for all input lengths:

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- Consider families $f_i\colon \{0,1\}^i \to \{0,1\}^i$.
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The most useful definition: $\{f_n\}_{n\in\mathbb{N}}$ s.t.

- $f_n\colon \{0,1\}^n \to \{0,1\}^n$ is computed in time $\mathrm{poly}(n)$.
- For every polynomial-time randomized adversary A,

$$\mathbf{Pr}\{A(f_n(x)) \text{ inverts } f_n\} \text{ is negligible}$$

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- For every polynomial-time randomized adversary A, for every k,

$$\mathbf{Pr}\{A(f_n(x)) \text{ inverts } f_n\} < \frac{1}{n^k}$$

for n big enough.

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$$\mathbf{Pr}\{A(f_n(x)) \text{ inverts } f_n\} \text{ is negligible}$$

- Here $\mathbf{Pr}$ is over $x \leftarrow U_n$ and A's randomness.
- For a non-uniform adversary, randomness does not matter.

Notation: negligible

Usually applied to probabilities

From now on:

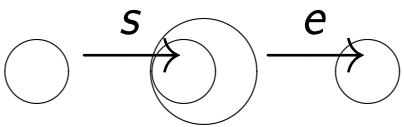
A sequence $t_1, t_2, \ldots$ is negligible
if for every k and every n large enough,

$$t_n < \frac{1}{n^k}.$$

It is non-negligible (inverse polynomial)
if there is k such that for every n large enough,

$$t_n \geqslant \frac{1}{n^k}.$$

Families of one-way functions (more generality)



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- $e_n\colon \{0,1\}^n \to \{0,1\}^{m(n)}$ (the function itself),
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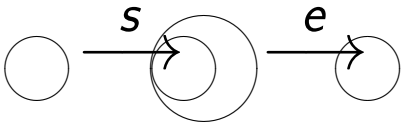

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such that

$\forall\, A$ (adversary, randomized polynomial-time)
$\forall k \in \mathbb{N}\ \ \exists N\ \ \forall n > N$

$$\mathbf{Pr}\{A(e_n(s_n(r_s)), 1^n, e_n, s_n) \in e_n^{-1}(e_n(s_n(r_s)))\} < \frac{1}{n^k},$$

where **Pr** is over A's randomness and uniformly distributed r_g, r_s.

Such an ugly and boring notation even at this level.
We will try to avoid it ☺.

Assumptions regarding lengths in one-way functions

Our owf's will be nice:

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- not necessarily defined on each length,
- can be sliced.

Exercise

The existence of

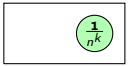
- one-way functions,
- owf with these nice "properties",
- families of one-way functions,

what is equivalent, what is not, what is unclear?

Note that a family has more than one pair (s_n, e_n) for each n.

"Weak" one-way functions

- "Strong" one-way functions: $\forall k$ an adversary cannot reach success probability $1/n^k$.



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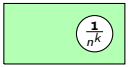

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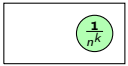

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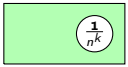

Weak $\mapsto$ Strong: *invert many copies*

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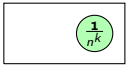

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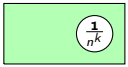

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Proof by reduction:

Assume that A_s inverts f_s with prob. $\geqslant 1/n^{k_s}$

$\Downarrow$

Construct A_w that breaks f_w.

$A'_w(y)$:

- `for each` j `in` $1..m$,
  - $x_1, \ldots, x_m \leftarrow$ random;
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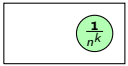

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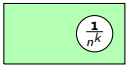

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$A'_w(y)$: *// Final A_w repeats it $a(n)$ times.*

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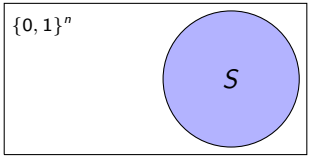

S is a "success set" for A'_w:

$$S = \left\{ y \in \{0,1\}^n \mid \mathbf{Pr}\{A'_w \text{ inverts } f_w \text{ in } y\} > \frac{n}{a(n)} \right\}$$

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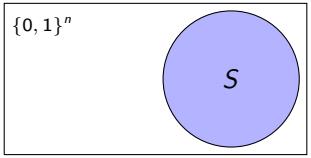

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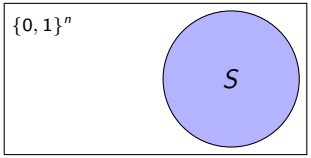

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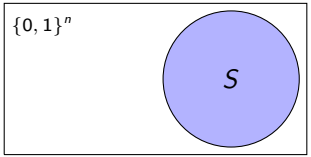

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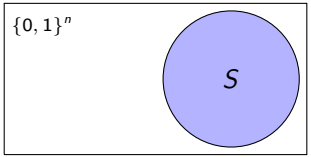

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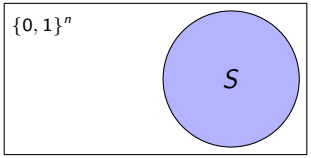

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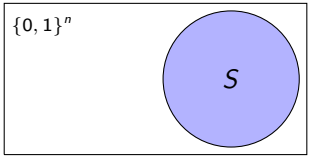

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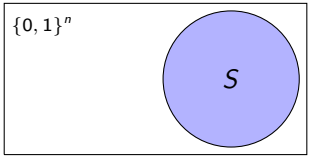

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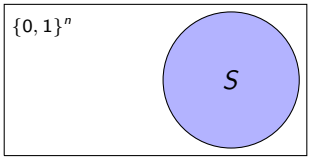

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Candidate one-way functions

Example (Not really)

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What do we mean by commas here?

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Example (Discrete logarithm)

Take $p \in \mathbb{P}$. Take g a generator in $\mathbb{F}_p$. Consider $x \in \mathbb{Z}_{p-1}$.

$f(x) = g^x \mod p$

Why "logarithm"? To invert, find " $\log_g(g^x)$ ".

Universal ("complete") one-way function

no complexity claim

$f \to g$, if $\exists$ (randomized) polynomial-time $T^\bullet$ $\forall k_f$ $\exists k_g$ $\forall$ adversary A

$$A \text{ inverts } g \text{ with prob. } 1 - \frac{1}{n^{k_g}} \implies T^A \text{ inverts } f \text{ with prob. } 1 - \frac{1}{n^{k_f}}.$$

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Theorem (the hardest-to-invert function)

There is a universal weak one-way function u, i.e., s.t. $\forall f \in \boldsymbol{f}\mathbf{P}$ $f \to u$.

algorithm, infinite storage

$u(M, x) = (M, M(x))$, where $x \in \{0,1\}^*$, M is the description of a Turing machine, $M(x)$ is the output of M on x *within* $|x|^2$ *steps (or just* $|x|$*).*

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Lemma

W.l.o.g. a weak owf is computable in $|x|^2$ steps.

$\tilde{f}(x_1x_2) = f(x_1)x_2$, where $|x_1| = n$, $|x_2| = m = m(n)$. Choose $m(n)$.

To break f:

- on input $y = f(x_1)$ concatenate a random x_2 to it;
- invert $y \cdot x_2$ using $\tilde{f}$;
- drop the suffix.

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Exercise

What about...

- strong owf?
- deterministic or non-uniform adversaries?

Administrivia

Grading scheme (not a cryptographic scheme)

Administrivia

Grading scheme

Suspense!

Administrivia

Grading scheme

- Homework exercises: 20%

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- Lecture notes: up to additional 10% (depending on the lecture)

Trapdoor functions

. . . come with their families

(first attempt)

Trapdoor function family (tdff) is a collection of polynomial-time computable functions e_n, d_n such that $d_n(e_n(x)) = x$ and for every adversary A,

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$\forall$ adversary A $\quad \forall k \in \mathbb{N} \;\; \exists N \;\; \forall n > N$

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Trapdoor function: example

Example (RSA, Rivest–Shamir–Adleman)

Take $p, q \in \mathbb{P}$. Let $N = p \cdot q$.

Pick ε, δ from $\mathbb{Z}_N$ s.t. $\varepsilon \cdot \delta \equiv 1 \pmod{(p-1)(q-1)}$.

Let

$$s(x) = x$$
$$e(x) = x^{\varepsilon} \mod N$$
$$d(x) = x^{\delta} \mod N$$

Discussion questions:

- why it works?
- give me p, q, ϵ, δ
- what if we know how to find factors efficiently

Takeout

- Weak and strong one-way functions
- Universal owf
- Trapdoor functions

Coming next: Godreich–Levin hardcore predicate, public-key encryption schemes