ARIEL UNIVERSITY

אוניברסיטת אריאל בשומרון

Foundations of Modern Cryptography

Edward A. Hirsch

https://edwardahirsch.github.io/edwardahirsch

Neapolis University Pafos

Lecture 3: October 17, 2024

Lecture Plan

Public-Key Encryption Schemes

- Constructions
- Definitions of security

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- Definitions of security

If the screen seems frozen and I do not respond,
please call me in Telegram.

Task: Public-Key Encryption

Alice
public key, secret key

Eve

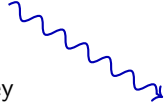
Bob
message

Task: Public-Key Encryption

public key

Alice

public key, secret key



Eve

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message

Task: Public-Key Encryption

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public key, secret key

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Pictures courtesy AI Emojis Generator

Task: Public-Key Encryption

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public key, secret key

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Eve

Eve should be unable to decrypt (not to say recover the secret key)
even though she has seen a lot of communication (and perhaps some messages).

Pictures courtesy AI Emojis Generator

Hardcore Predicate

- One-way functions are hard to invert.
- What if we want to recover a *single* (say, the first) bit of the argument? Is it still hard?

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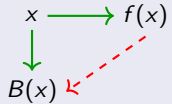
Definition

$B : \{0,1\}^* \to \{0,1\}$ is a hardcore predicate for $f : \{0,1\}^* \to \{0,1\}^*$ if
no adversary can guess $B(x)$ from $f(x)$ with probability non-negligibly better than 50%



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A family of polynomial-time computable functions
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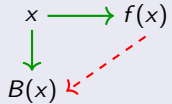

$$\forall k\ \forall A\ \exists N\ \forall n > N \quad \mathbf{Pr}\{A(f_n(x)) = B_n(x)\} < \frac{1}{2} + \frac{1}{n^k},$$

where A is a randomized polynomial-time adversary;
the probability is taken over $x \in U_n$ and over the randomness used by A.

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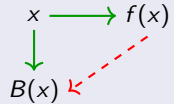

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Theorem (Oded Goldreich, Leonid Levin)

If f is a strongly one-way function, then $\tilde{f}(x,r) = (f(x), r)$ is a strongly one-way function and $B(x,r) = \langle x, r\rangle \bmod 2 = x_1 r_1 \oplus x_2 r_2 \oplus \ldots$ is its hardcore predicate. *// Proof delayed.*

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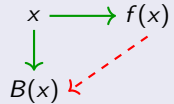

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Exercise

The construction works for trapdoor permutation families as well. Formulate and prove this.

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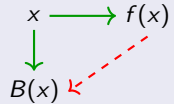

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Problem

This theorem does not claim that there is a hardcore predicate for f. Is it possible to build one?

Public-Key Encryption Scheme

All algorithms below take time polynomial in n.
They can be randomized (and thus make errors) or deterministic.

INGREDIENTS:

- G (Alice) generates a pair of keys (public key pk, secret/private key sk):

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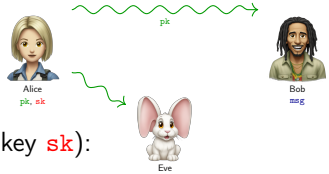

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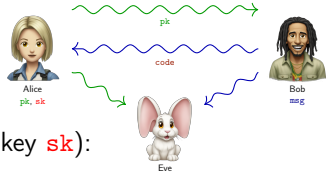

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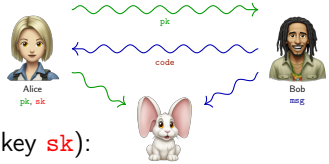

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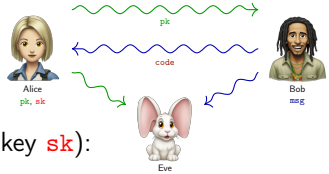

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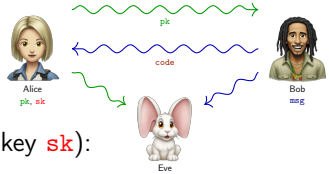

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  or at least

$$\mathbf{Pr}\{D(\mathtt{sk}, E(\mathtt{pk}, \mathtt{msg})) = \mathtt{msg}\} \geqslant 0.9.$$

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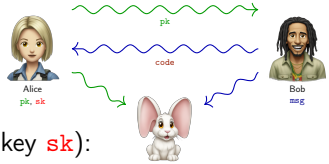

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- An adversary $A(\mathtt{pk}, \mathtt{code})$ (Eve) should not be able to decrypt (without $\mathtt{sk}$) with non-negligible probability, at least

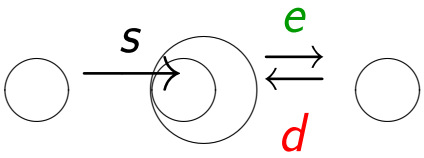
$$\mathbf{Pr}\{A(\mathtt{pk}, E(\mathtt{pk}, \mathtt{msg})) = \mathtt{msg}\} < \varepsilon(n).$$

  negligible

  (more on this later).

Public-Key Encryption Scheme

A construction for encrypting 1 bit based on trapdoor permutations and hardcore predicates



Caution: discrepancy in the literature what is "permutation"

Assume that G generates a trapdoor permutation family:
deterministic polynomial-time $G\colon (1^n, r_g) \mapsto (e, d, s)$ (Boolean circuits) s.t.

- $e\colon \{0,1\}^n \to \{0,1\}^{\zeta(n)}$
- $d\colon \{0,1\}^{\zeta(n)} \to \{0,1\}^n$
- $s\colon \{0,1\}^{\sigma(n)} \to \{0,1\}^n$

such that $\forall x \in \mathrm{Im}(s)\ d(e(x)) = x$.

Let $B(x)$ be a hardcore predicate, $B(x)$ is hard to guess from $e(x)$.

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If A breaks it, we break the hardcore predicate:

Take $b \in \{0, 1\}$ at random.
Compute $B(x)$ from y as $\beta := A(\mathtt{pk}, (y, b)) \oplus b$ $/\!/ x = s(r)$, $y = e(x)$

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$$

Distributions: $x \leftarrow s(r)$, $r \leftarrow U_{s(n)}$, $b \leftarrow U_2$, $r_g \leftarrow U_{\ldots}$; A, D, E are randomized.

Exercise

It is hard to decrypt a *random* message (bit). Is it enough?

Is it secure enough?

A TCS approach: Computational indistinguishability

How to distinguish...

- a Geiger counter output from a general-purpose computer output,

Who tries to distinguish?

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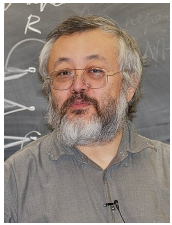

Alexander Shen

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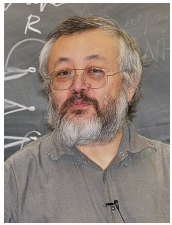

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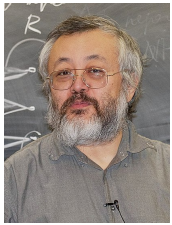

Alexander Shen

Taken by A. Smal

typically polynomial-time

Definition

Probability distributions P and Q are computationally indistinguishable if $\forall$ adversary A // *Outputs 0 or 1.*

$$\left|\mathbf{Pr}_{x\leftarrow \mathsf{P}}\{A(x)=1\} - \mathbf{Pr}_{x\leftarrow \mathsf{Q}}\{A(x)=1\}\right| < \varepsilon(n)$$

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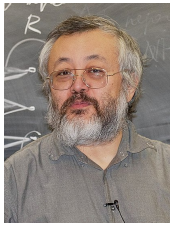

Alexander Shen

Taken by A. Smal

typically polynomial-time

Definition

Probability distributions P and Q are computationally indistinguishable if $\forall k$ $\forall$ adversary A // *Outputs 0 or 1.*

$$\left|\mathbf{Pr}_{x \leftarrow P_n}\{A(x) = 1\} - \mathbf{Pr}_{x \leftarrow Q_n}\{A(x) = 1\}\right| < \frac{1}{n^k}$$

for n big enough ($P = \{P_n\}_n$, $Q = \{Q_n\}_n$ are ensembles of distributions).

Indistinguishability as a security criterion (not the only one!)

Not just single-bit messages: their bits are dependent

PKES = PKCS

Definition

or generated by the adversary?

A PKCS is computationally indistinguishable if $\forall$ messages (m_0, m_1) of polynomial length $\forall$ adversary A

$$\Big| \; \mathbf{Pr}\{A(E(m_0, e, r_e), e, 1^n, m_0, m_1) = 1\} \; - \; \mathbf{Pr}\{A(E(m_1, e, r_e), e, 1^n, m_0, m_1) = 1\} \; \Big| \; < \varepsilon(n).$$

The probability is taken over r_g, r_e, and A's randomness.

Indistinguishability of 1-bit PKCS

Theorem (already proved)

For the 1-bit PKCS we constructed, $\forall A$

$$\mathbf{Pr}\{A(e(\mathtt{msg}, r_e), 1^n, e) = \mathtt{msg}\} < \frac{1}{2} + \varepsilon(n),$$

where $G(1^n, r_g) = (e, d)$, and **Pr** is over everything including msg.

Definition

1-bit PKCS is computationally indistinguishable, if $\forall A$

$$\left|\mathbf{Pr}\{A(e(1, r_e), 1^n, e) = 1\} - \mathbf{Pr}\{A(e(0, r_e), 1^n, e) = 1\}\right| < \varepsilon(n),$$

where $G(1^n, r_g) = (e, d)$, and **Pr** is over everything excluding msg.

Exercise (do it now?)

Is our PKCS indistinguishable?

Public-Key Encryption Scheme for arbitrary messages

Let $E^*(b_1 b_2 \ldots) = (E(b_1), E(b_2) \ldots)$ (the same key pair!).

Break the old (1-bit) cryptosystem using an adversary A^* for this one.

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A^* can distinguish two messages $m_0 = s_1 \ldots s_p$ and $m_1 = t_1 \ldots t_p$,

$\frac{1}{n^k} < |\mathbf{Pr}\{A^*(E(s_1 s_2 \ldots s_{p-1} s_p \ldots) = 1\} - \mathbf{Pr}\{A^*(E(t_1 t_2 \ldots t_{p-1} t_p \ldots) = 1\}| \leqslant$

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&\qquad \ldots \\
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\end{aligned}$$

There is a substantial difference somewhere here: $|\ldots s_{i_*} \ldots - \ldots t_{i_*} \ldots| > \frac{1}{n^k}\frac{1}{p} \geqslant \frac{1}{n^r}$.

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\end{aligned}$$

There is a substantial difference somewhere here: $|\ldots s_{i_*} \ldots - \ldots t_{i_*} \ldots| > \frac{1}{n^k}\frac{1}{p} \geqslant \frac{1}{n^{k'}}$.

Substitute $\texttt{code}_a$ for s_i and $\texttt{code}_b$ for t_i, generate other bits s_j, t_j at random.

Where is i_*? Try a random i. Overall success:

$$\mathbf{Pr}\{i = i_*\} \cdot \frac{1}{n^{k'}}.$$

$\mathbf{Pr}\{i = i_*\} = \frac{1}{p}$ is good if the messages were of polynomial length.

Is it secure enough?

Behavioural approach: Semantic security

Definition

A PKCS is semantically secure if $\forall f\ \forall h\ \forall A\ \forall M$

$$\mathbf{Pr}\{A(E(e,m),e,h(m)) = f(m)\} \leqslant \mathbf{Pr}\{\widetilde{A}(e,h(m)) = f(m)\} + \varepsilon(n),$$

where the hint h and the "sense" f are polynomial-time computable,
M is a message generator,
A is an adversary trying to guess f from `code`,
$\widetilde{A}$ is an "adversary" guessing f without `code`.

Alice

pk, sk

Bob

msg

Eve

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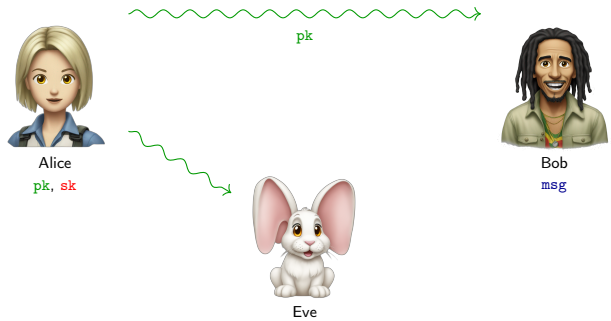

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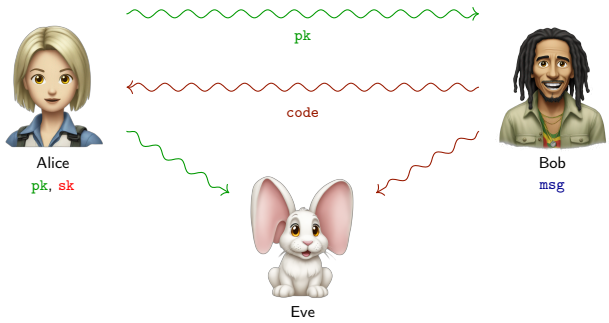

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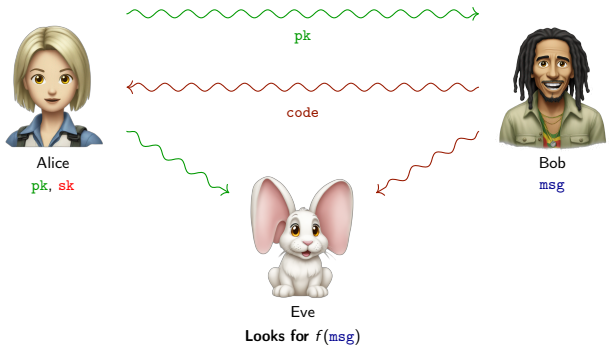

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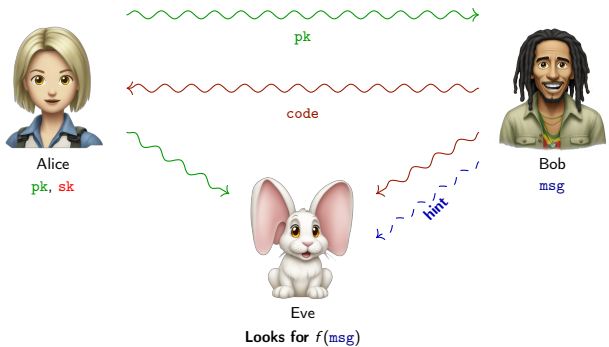

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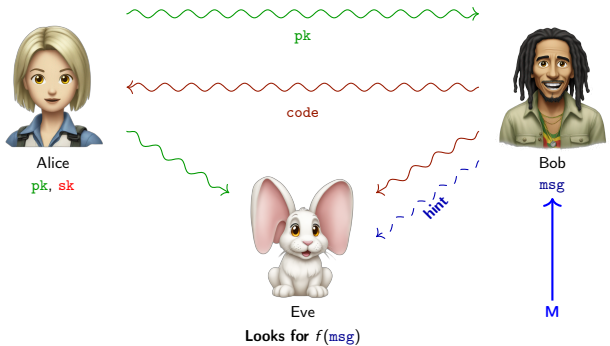

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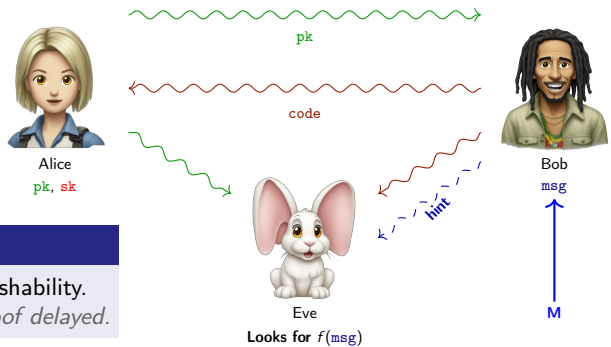

Theorem

Semantic security $\Leftrightarrow$ computational indistinguishability.

// Proof delayed.

A more efficient PKCS

- $E^*(b_1 b_2 \ldots) = (E(b_1), E(b_2) \ldots)$ has a separate codeword for each bit.

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- r is short for $s(r)$ (a random string generated by the sampler)
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- $E^{**}(b_1 \ldots b_m, e, r) = (e^m(r), B(r) \oplus b_1, B(e(r)) \oplus b_2, \ldots)$, is better.
- We will later see that its codewords are indistinguishable from codewords of random messages.

Takeout

- PKCS from a trapdoor permutation and a hardcore predicate.
- Indistinguishability vs security.

Coming next:

- *Goldreich–Levin Theorem proof.*
- *Indistinguishability vs security proof.*
- *PRGs (pseudorandom generators).*