ARIEL UNIVERSITY

אוניברסיטת אריאל בשומרון

Foundations of Modern Cryptography

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Lecture 5: October 31, 2024

Lecture Plan

- Indistinguishability implies semantic security — a proof.
- Private-key cryptography: not so simple.

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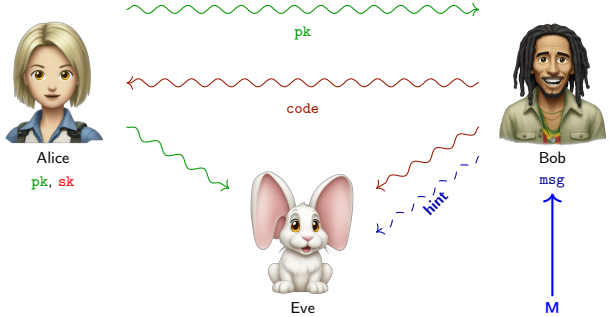
- Indistinguishability implies semantic security — a proof.
- Private-key cryptography: not so simple.

If the screen seems frozen and I do not respond,
please call me in Telegram.

Computational indistinguishability vs Semantic security



Definition

A PKCS is semantically secure if $\forall f\ \forall h\ \forall A\ \forall M$

$$\mathbf{Pr}\{A(E(e,m),e,h(m)) = f(m)\} \leqslant \mathbf{Pr}\{\widetilde{A}(e,h(m)) = f(m)\} + \varepsilon(n),$$

where the hint h and the "sense" f are ~~polynomial-time computable~~ (we'll see),
M is a message generator, A is an adversary trying to guess f from `code` (ciphertext),
$\widetilde{A}$ is an "adversary" guessing f without `code`.

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Probability distributions P and Q are computationally indistinguishable if $\forall$ adversary A

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or generated by the adversary?

A PKCS is computationally indistinguishable if $\forall$ messages (m_0, m_1) of polynomial length $\forall$ adversary A

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The probability is taken over r_g, r_e, and A's randomness.

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Semantic security $\Leftrightarrow$ computational indistinguishability *// It remains to prove $\Leftarrow$*

Indistinguishability to semantic security (1)

- Assume A_S can compute $f(m)$, then construct m_0, m_1 with $f(m_0) \neq f(m_1)$ that we will distinguish.
- New adversary A_I given encrypted m distinguishes $m = m_0, m_1$ sent with prob. $\frac{1}{2}$ each:
  `if` $A_S(E(m, \ldots), h(m_0)) = f(m_0)$ `then return 0,`
  `otherwise flip a fair coin and return the result.`
- Note that $f(m_0), h(m_0)$ are **hardwired into the circuit** A_I.

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- Note that $f(m_0), h(m_0)$ are **hardwired into the circuit** A_I.
- Let
  - $p_k(x) := \mathbf{Pr}\{A_S(E(x,\ldots), h(m)) = k\}$, *// Normal scenario for A_S*
  - $q_k(x) := \mathbf{Pr}\{A_S(E(x,\ldots), h(m_0)) = k\}$, *// Wrong hint, what would it say?*
  - $f_i := f(m_i)$.
- $\mathbf{Pr}\{A_I \text{ succeeds}\} =$

$$\mathbf{Pr}\{m_0 \text{ is sent}\} \cdot \left(p_{f_0}(m_0) + \frac{1}{2}(1 - p_{f_0}(m_0))\right) + \mathbf{Pr}\{m_1 \text{ is sent}\} \cdot \left(\frac{1}{2}(1 - q_{f_0}(m_1))\right) =$$

$$\frac{1}{2}\left(1 + \frac{p_{f_0}(m_0) - q_{f_0}(m_1)}{2}\right)$$

- It remains to prove that there are m_0, m_1 with substantial $p_{f_0}(m_0) - q_{f_0}(m_1)$.

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$$\sum_{m_0, m_1} p(m_0)p(m_1)(p_{f_0}(m_0) - q_{f_0}(m_1)) \overset{\sum_{m_1} p(m_1)=1}{=\!=\!=} \sum_x p(x)p_{f(x)}(x) - \sum_{m_0, m_1} p(m_0)p(m_1)q_{f_0}(m_1). \qquad (*)$$

The first part is the prob. of success of the original A_S.

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Let $F_k = \{x | f(x) = k\}$ and $\mu_k = \mathbf{Pr}\{M_S(1^n) \in F_k\}$, then split the second part by F_k:

$$\sum_k \sum_{m_0 \in F_k} \sum_{m_1} p(m_0)p(m_1)q_k(m_1) = \sum_k \left(\left(\sum_{m_1} p(m_1)q_k(m_1) \right) \sum_{m_0 \in F_k} p(m_0) \right) = \sum_k \left(\mu_k \left(\sum_{m_1} p(m_1)q_k(m_1) \right) \right)$$

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. . . and it is the success prob. (on random $m_0 \leftarrow M_S$) for the following oblivious $\tilde{A}_S(h_*)$, where $h_* = h(m_0)$:

take random $m \leftarrow M_S$
encrypt m, run $A_S(E(m, \ldots), h_*)$

Thus (*) is small, contradiction.

Remark (a stronger definition)

We can assume that the oblivious adversary's circuit $\tilde{A}_S$ is generated from the normal adversary's circuit A_S in polynomial time.

Exercise

There are two more things that we can change in the definitions:

- Uniform (algorithms) vs non-uniform (circuits) adversaries (including samplers).
- Messages are generated by an adversary or the system should be secure for every two m_0, m_1.

How do they change the implications ($\Rightarrow$ and $\Leftarrow$) or their proof?

Exercise

We did not say whether the oblivious $\tilde{A}_S$ should get also the public key or not. Is it important?

Remark

Our definitions have been the same for public- and private-key settings so far.

Is it enough for our security?

Multiple messages

Attempted attack:

- E has been used for two messages 0^t and 1^t

$$\begin{array}{rcccc} E_x(0^t) & = & (f^t(x), & B(x) \oplus 0, & B(f(x)) \oplus 0, \quad \ldots) \\ E_y(1^t) & = & (f^t(y), & B(y) \oplus 1, & B(f(y)) \oplus 1, \quad \ldots) \end{array}$$

Any information can be extracted from $E_x(0^t) \oplus E_y(1^t)$?

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- For our PKCS, no. If someone can distinguish sequences,

$$E(m_1), \quad E(m_2), \quad \ldots, \quad E(m_{i^*}), \quad \ldots, \quad E(m_k)$$

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then one can distinguish sequences of almost similar messages, there is a crucial transition point i^*.

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- Now generate everything ourselves (using the public key) and distinguish $E(???_1), E(???_2)$ given to us.

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- Now generate everything ourselves (using the public key) and distinguish $E(???_1), E(???_2)$ given to us.
- True for public-key encryption only (Eve can't encrypt in the private-key setting).

Private-key (Symmetric-key) encryption

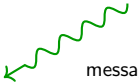
Alice

shared secret key

encrypted message(s)

Bob

message(s), shared secret key



Eve

Definition

Indistinguishable or Semantically secure

Symmetric-key encryption scheme is secure in the multiple-messages setting if

[. . . all the same words as for a single message but . . .]

an adversary is given a sequence of messages (generated by M_S) encrypted with the same key

[. . . and is asked to break them all: f and h depend on all the messages]

Symmetric-key cryptography using owp / PRG: How NOT to do it

. . . and why it is important in practice

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- Symmetric key `sk`: a seed for a (publicly known) PRG G.

$$E(\texttt{sk}, \texttt{msg}) = \texttt{msg} \oplus G(\texttt{sk}) = (b_1 \oplus G_1(\texttt{sk})) \circ (b_2 \oplus G_2(\texttt{sk})) \circ \ldots \circ (b_p \oplus G_p(\texttt{sk})),$$

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- Easy to break — complete the details (exercise).

Exercise

Show formally that this system is not semantically secure for multiple messages.

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- If messages are lost, it takes proportional time to recover s' by repeating G.

Exercise

Prove that this scheme is secure.

Pseudorandom functions

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- Your construction should not distinguish it as well $\rightarrow$ you can use f_ζ instead.

Pseudorandom functions

Definition (Pseudorandom function family, prff)

A prff is a family of functions $\{f_\zeta\}_\zeta$, where $f_\zeta \colon \{0,1\}^n \to \{0,1\}^n$, computed by Boolean circuits generated by a polynomial-time computable algorithm $Z \colon (1^n, r_Z) \mapsto f_\zeta$ such that f_ζ is computationally indistinguishable from a random function: for any adversary A,

$$|\mathbf{Pr}_{A,y}\{A(y,1^n) = 1 \mid y \leftarrow f_\zeta(U_n)\} - \mathbf{Pr}_{A,R,x}\{A(x,1^n) = 1 \mid x \leftarrow R(U_n)\}| < \varepsilon(n),$$

where $R \colon \{0,1\}^n \to \{0,1\}^n$ has a uniformly random truth table.

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Construction from PRGs

- One can construct a prff using a $2n$-PRG $G \colon \{0,1\}^n \to \{0,1\}^{2n}$.
- Split G's output into two n-bit parts $G(x) = G_0(x) \circ G_1(x)$.
- Take $s \leftarrow U_n$, compute
$$\zeta_s(b_1 b_2 \dots b_n) = G_{b_n}(G_{b_{n-1}}(\dots(G_{b_1}(s)\dots)).$$
- (Proof delayed.)

A standard definition says that Z generates an index, and $f_\zeta(x)$ is computed by a polynomial-time algorithm $F(\zeta, x)$, observe that it is equivalent to our definition.

A symmetric-key encryption scheme

Based on prff

- To encode an n-bit message, define a "proto-scheme"

$$E(\zeta, \texttt{msg}, r_E) = (\zeta(r_E) \oplus \texttt{msg}, r_E),$$

where $\zeta \leftarrow Z(1^n)$ serves as a key and $r_E \leftarrow U$ is our randomness.

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- To encode messages of arbitrary length, cut them into pieces:

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- If an adversary distinguishes

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then we can distinguish either

$$\zeta \text{ and } R$$

or

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The latter two are indistinguishable: with prob. $\geqslant 1 - t^2 \cdot 2^{-n}$ all r_i's are distinct and thus $R(r_i) \oplus \ldots$ are distributed uniformly and independently.

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- Untrusted channel is a problem for public-key cryptography as well (requires certification).

Possible practical (not purely mathematical) attacks on encryption

Unencrypted message

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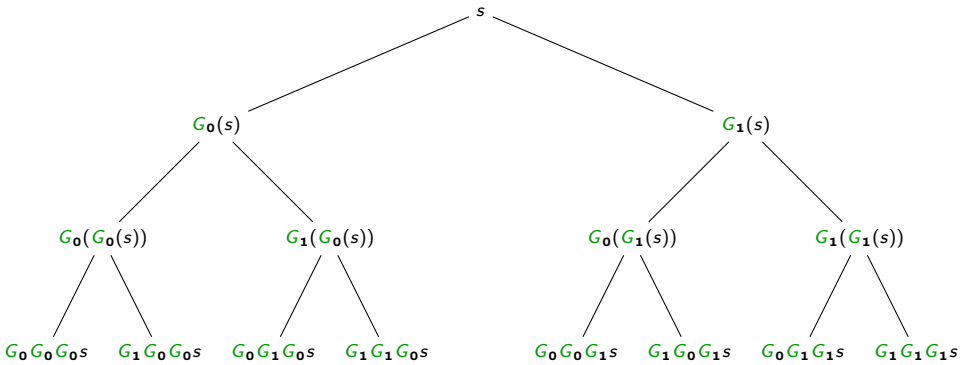
- More practical issues: leakage due to interference with other programs.

Prff from PRG: Why it works

Distinguish $(x_1^\ell, x_1^r), (x_2^\ell, x_2^r), \ldots, (x_p^\ell, x_p^r)$ from truly random source using a distinguisher D of $\zeta_s(b_1 b_2 \ldots b_n) = G_{b_n}(G_{b_{n-1}}(\ldots(G_{b_2}(G_{b_1}(\zeta))\ldots))$ from $R(b_1 b_2 \ldots b_n)$



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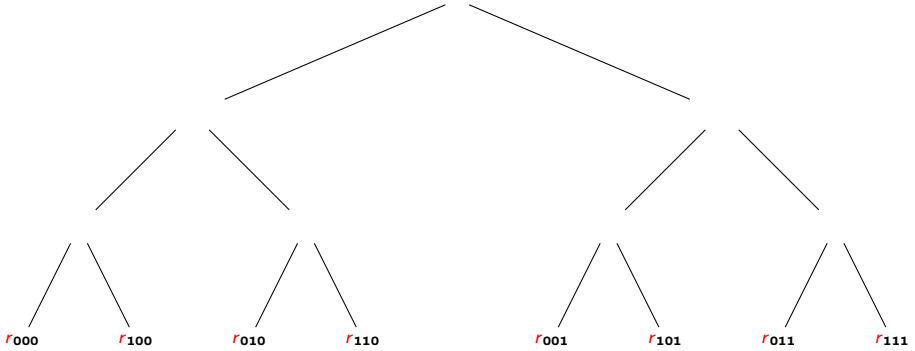

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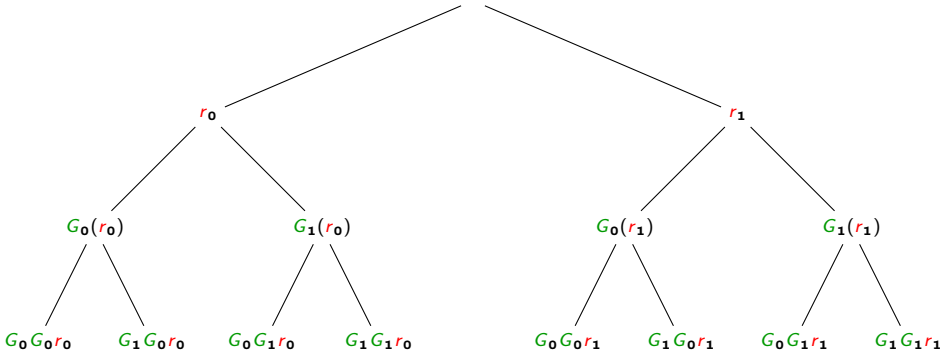

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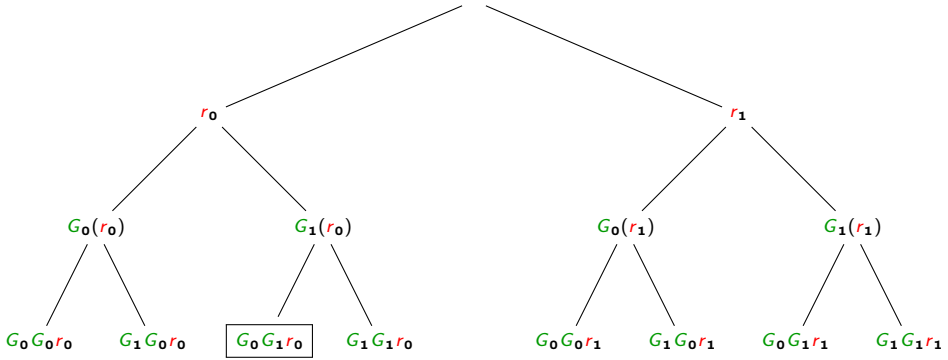
Suppose D asks for $\boxed{f(010)}\ldots$



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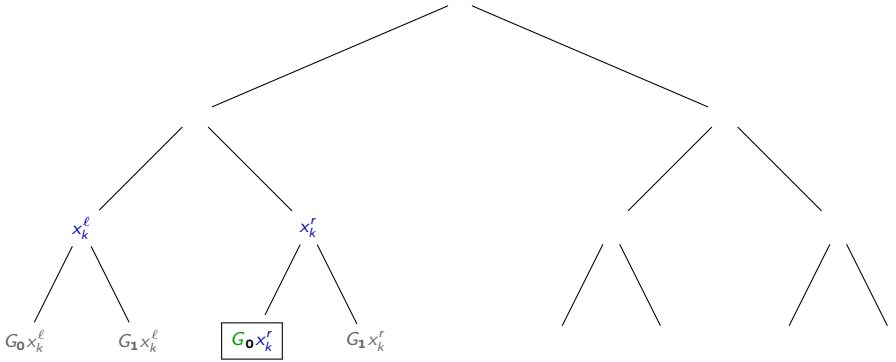

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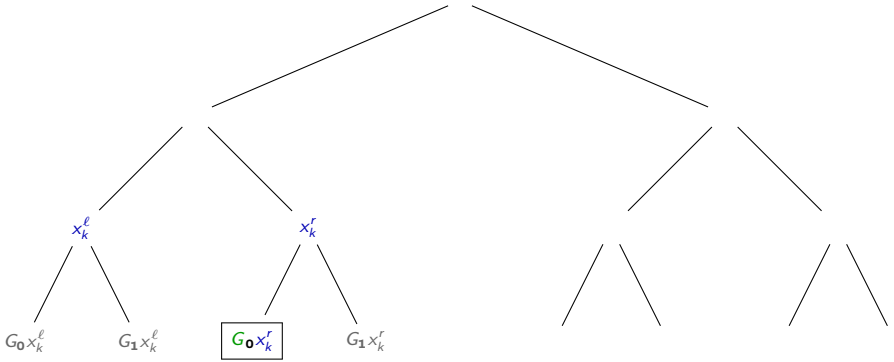

Note that a PRG stays secure if queried polynomially many times.

Takeout

- Indistinguishability implies semantic security — a proof.
- Symmetric-key encryption based on pseudorandom function families.
- Diffie–Hellman key-exchange protocol.
- Prff from PRG.

Coming next:

- *Digital signatures.*