ARIEL UNIVERSITY

אוניברסיטת אריאל בשומרון

Foundations of Modern Cryptography

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Neapolis University Pafos

Lecture 8: December 5, 2024

Lecture Plan

- Zero-knowledge proofs.

If the screen seems frozen and I do not respond,
please call me in Telegram.

Task: Zero-Knowledge Proof

Alice

x, proof that $x \in L$

Bob

x

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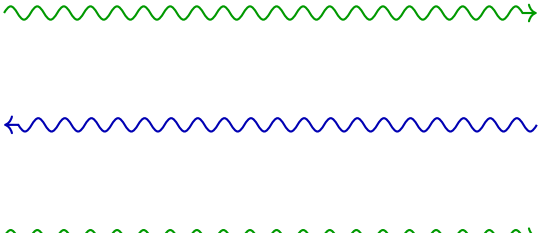

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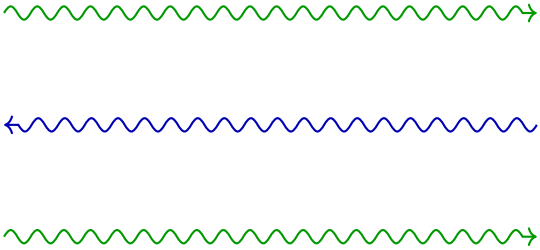

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Pictures courtesy AI Emojis Generator

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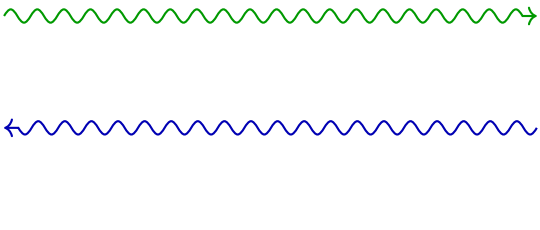

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Bob gets convinced iff $x \in L$ (with certain probability of error).

Bob does not learn from this protocol about Alice's proof more than the fact $x \in L$.

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Strong ZK: Bob also gets convinced that Alice indeed knows some proof that Bob can verify.

Pictures courtesy AI Emojis Generator

Zero-Knowledge Proofs: Definition

Definition

Language L has a computational zero-knowledge proof ($L \in \mathbf{CZK}$) if there is a protocol between prover $P(x)$ and polynomial-time bounded randomized verifier $V(x)$ s.t.

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(CZK) and for every p.-t. b.r. V' (outputting a lot!) there is an oblivious $\tilde{V}$ s.t. $\forall x \in L$ the distributions

$$\{V'_r(x, \text{protocol}(P, V'_r)(x))\}_{r \in U} \text{ and } \{\tilde{V}_t(x)\}_{t \in U}$$

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Remark

There are also other classes: *statistical* and *perfect* zero-knowledge. We don't need them.

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Remark

It is also a strong CZK proof: given access to P (for fixed G, c) as an oracle, one can reconstruct (some) coloring in polynomial time, thus P "knows" a coloring.

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- If it does not fail, it's results (even transcripts) are indistinguishable from that of V'.
  - Intuitively clear, and you don't want to see the formal proof.

Takeout

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Coming next:

- *How ZK helps to force malicious participants to be semi-honest.*
- *(maybe) Exercises and Repeat.*
- *Blockchains.*