ON A CONSTRUCTION OF SYMBOLIC REALIZATION OF HYPERBOLIC AUTOMORPHISMS OF TORUS

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In the paper [1] A. M. Vershik suggested a general approach to constructing of arithmetic isomorphism of hyperbolic automorphisms of torus and symbolic shifts. An initial step of suggested scheme consisted in the following conjecture.

Let T be an automorphism of torus; we shall use the same notation for the corresponding transformation of $\mathbb{Z}^n$. Let a vector $v \in \mathbb{Z}^n$ have the property that its orbit under the automorphism T in $\mathbb{Z}^n$ is infinite. Let us denote by G the semigroup generated by the orbit of the element v under T.

Conjecture. *There is a natural number N such that every element g of the semigroup G can be represented in the following way:*

$$g = \sum_{k \in \mathbb{Z}} e_k(g)\, T^k v\,,$$

where $e_k(g)$ is a finite sequence of numbers $0, 1, \dots, N$.

There exists a proof of this conjecture in case when the characteristic polynomial of T is of the form $x^n - a_{n-1}x^{n-1} - \dots - a_1 x_1 - 1$, where $a_{n-1} \geqslant \dots \geqslant a_2 \geqslant a_1 \geqslant 0$ (see [2]). A similiar approach was realized in [3] for an automorphism such that the dominant root of the characteristic polynomial of it is a Pisot number. (But instead of the conjecture under discussion a slightly different statement was used.) A. M. Vershik suggested a hypothesis that this result can be extended to a wider class of automorphisms. * In this note we prove that it is impossible to extend it to one class of automorphisms. Namely, we consider automorphisms, characteristic polynomials of which have nonnegative coefficients and at least two roots of different modulus outside the unit

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* As A. M. Vershik informed the author, he together with R. Kenyon proved that the Conjecture will be true if one replace the condition $e_k(g) \in \{0, 1, \dots, N\}$ with the condition $e_k(g) \in S$, where S is a finite subset of the Galois field of the characteristic polynomial of the automorphism T.

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circle. Thus in case of nonnegative coefficients two opportunities remain unconsidered:

(1) if there are roots on the unit circle;
(2) if all roots outside the unit circle have the same modulus.

So let us consider a polynomial $p(z) = z^n - a_{n-1}z^{n-1} - \ldots - a_1 z_1 - 1$, where $a_{n-1}, a_{n-2}, \ldots, a_1 \geqslant 0$, $\sum a_i^2 > 0$.

Let S be the set of all two-sided finite sequences $(\ldots, b_{-2}, b_{-1}, b_0, b_1, b_2, \ldots)$ of nonnegative integers. Elements of S will be regarded as formal sums of the form

$$\sum_{i=s_1}^{s_2} b_i\, z^i \quad (s_2 \geqslant s_1).$$

Let $v \in S$. We denote the largest of its coefficients by $rk\, v$.

Let $v, w \in S$. We write $v \backsim_p w$, if $\exists\, t \in \mathbb{Z} : p \mid (v - w)\, z^t$.

We call p **decomposing**, if

$$\exists C \in \mathbb{N}\ \forall v \in S\ \forall w \in S\ \ (v \backsim_p w\ \ \&\ \ rk\, w \leqslant C).$$

We note that p has exactly one positive root $r > 1$.

Theorem. *If the polynomial p satisfying the conditions above has a (complex) root q which is situated outside the unit disk and differs by modulus from r then p is not decomposing.*

Proof. Suppose the assertion does not hold, i.e. p is decomposing. Then

$$\exists C \in \mathbb{N}\ \forall v \in S\ \forall w \in S\ \ (v \backsim_p w\ \ \&\ \ rk\, w \leqslant C).$$

Take $v = A$ (constant). $w \backsim_p A$, hence $w(r) = w(q) = A$. Let

$$w = \sum_{i=s_1}^{s_2} b_i z^i, \quad b_{s_2} > 0.$$

Then $A = w(r) \geqslant r^{s_2}$, i.e. $s_2 \leqslant \log_r A$.

From the other hand,

$$|A| = |w(q)| < C\, \frac{|q|^{s_2+1}}{|q| - 1},$$

i.e.

$$s_2 > \log_{|q|} A + \log_{|q|} \frac{|q| - 1}{C} - 1.$$

The value of $\log_{|q|} \frac{|q|-1}{C} - 1$ is a constant not depending on A; let us denote it by D.

So

$$\log_r A \geqslant s_2 > \log_{|q|} A + D. \qquad (*)$$

However, notice that $r > |q|$, since otherwise, i.e. if $r < |q|$, we have

$$0 = \frac{p(q)}{(q)^n} = 1 - \sum_{i=1}^{n-1} a_i\, q^{-i} - q^{-n} \geqslant$$

$$\geqslant 1 - \sum_{i=1}^{n-1} a_i\, |q|^{-i} - |q|^{-n} = \frac{p(|q|)}{|q|^n} > \frac{p(r)}{|q|^n} = 0$$

(contradiction).

Thus for sufficiently large A inequality $(*)$ does not hold and this contradicts to the initial hypothesis that p is decomposing

References

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